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TURBULENT DIFFUSION OF A PASSIVE IMPURITY*

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A solution of the diffusion equation in the case of a medium which is diffusing in an inhomogeneous and non-stationary manner is constructed using the Feynman operator formalism. The functional transformation proposed by Stratonovich is used for the "disentanglement of the operator exponent". As a result, the solution is represented in the form of a continual integral which differs from that obtained by Wiener in that, instead of an integral along trajectories, an integral of the velocities of the motion along the trajectories occurs in it. A statistical solution of the diffusion equation is obtained after averaging over random velocities. In the case of Gaussian statistics for the velocity fields or in the case of a spatially homogeneous non-stationary velocity field, continual integration can be carried out in an explicit form in the Markov approximation. In the first case, the result reduces to a renormalization of the coefficient of viscosity (the replacement of the coefficient of molecular viscosity by an effective coefficient of viscosity) and, in the second case, to the replacement of real time by an effective time. A number of papers, a list of which can be found /1/, are concerned with the problem of finding a technique for the "summation" of the coefficients of molecular and turbulent transport (to be specific, we shall speak about diffusion).

1. The equation which describes the propagation of a passive impurity in the velocity pulsation field has the form /1/

$$[\partial_t + \mathbf{v}(\mathbf{r}, t)\,\partial - \varkappa\partial^2]\, C(\mathbf{r}, t) = 0 \tag{1.1}$$

where $C(\mathbf{r}, t)$ is the concentration of the impurity, $\mathbf{v}(\mathbf{r}, t)$ is the turbulent velocity and $\varkappa$ is the molecular transport coefficient.

The diffusion coefficient can be determined in various ways /2-4/ but it is most natural to associate it with the rate of spreading of the initially localized impurity concentration distribution /3/. By virtue of the linearity of Eq.(1.1), the solution of the corresponding Cauchy problem is defined in terms of Green's function, which describes the response of the concentration $C(\mathbf{r}, t)$ to a change in the density of the passive impurity source $\rho(\mathbf{r}', t')$

$$\bar{G} = \langle G(\mathbf{r}, t; \mathbf{r}', t' \mid \mathbf{v}(\mathbf{r}, t))\rangle = \delta\langle C(\mathbf{r}, t)\rangle / \delta\rho(\mathbf{r}', t') \tag{1.2}$$

where the angular brackets denote averaging over the ensemble of samples of the velocity field $\mathbf{v}(\mathbf{r}, t)$. Green's function for an individual sample of the concentration field is a functional of the individual sample of the velocity field and satisfies the equation

$$\begin{gathered} L(\partial_t, \partial \mid \mathbf{v}(\mathbf{r}, t))\, G(\mathbf{r}, t;\ \mathbf{r}', t' \mid \mathbf{v}(\mathbf{r}, t)) = \\ [\partial_t + \mathbf{v}(\mathbf{r}, t)\,\partial - \varkappa\partial^2]\, G(\mathbf{r}, t;\ \mathbf{r}', t' \mid \mathbf{v}(\mathbf{r}, t)) = \delta(\mathbf{r} - \mathbf{r}')\,\delta(t - t') \end{gathered} \tag{1.3}$$

If the random velocity field is homogeneous and stationary, the function $\bar{G}$ depends solely on the difference in the coordinates and times and, in the case of this function, it is possible to carry out a Fourier transformation and to conduct the treatment in the space of the Fourier images. It can be shown by the methods of quantum field theory /3, 5/ that the mean Green's function can be represented in the form

$$\bar{G}(\mathbf{p}, \omega) = [-i\omega + \varkappa^*(\mathbf{p}, \omega)\, p^2]^{-1} \tag{1.4}$$

where $\varkappa^*(\mathbf{p}, \omega)$ should be considered as an effective diffusion coefficient which describes the transport of a passive impurity by turbulent velocity pulses.

Unfortunately, it is not possible to write the equation for G in closed form, since the equation for the concentration is non-linear with respect to the set of random fields $C(\mathbf{r}, t)$ and $\mathbf{v}(\mathbf{r}, t)$, as a consequence of which the equation for the statistical moment of order n will also contain moments of order $n + 1$ even when the velocity field statistics are Gaussian. This is the difference between multiplicative noise (the coefficients of the equation are

*Prikl. Matem. Mekhan., 55, 2, 275-282, 1991 random functions) and the corresponding Langevin approach of additive noise (the right-hand sides of the equations are random functions): in the case of additive noise, Gaussian statistics of the external perturbations lead, in a linear system, to Gaussian statistics for the quantities being considered. The chain of equations for the moments can be broken by requiring that the semi-invariants (of the cumulative means) should vanish starting from a certain order (the hypothesis of quasinormality) and while there is no rigorous proof of such a procedure, the deviations from Gaussian statistics do not have to be small.

2. The sole remaining possibility for calculating the function $\bar{G}(\mathbf{r}, t)$ involves finding $G(\mathbf{r}, t;\ \mathbf{r}', t' \mid \mathbf{v}(\mathbf{r}, t))$ at a specified velocity and subsequent averaging over the velocity ensemble in accordance with formula (1.2). However, the complexity of Eq.(1.3) lies in the fact that its coefficients are variable random functions.

One can attempt to seek a solution using the methods of perturbation theory in the form of a functional series in powers of $\mathbf{v}(\mathbf{r}, t)$ by considering a term which is proportional to the velocity as the perturbation /6/. In order to do this, we represent Eq.(1.3) in the integral form

$$G(\mathbf{r}, t; \mathbf{r}', t' \mid \mathbf{v}(\mathbf{r}, t)) = G^{(0)}(\mathbf{r}, t; \mathbf{r}', t') - \int d\mathbf{r}'' dt'' G^{(0)}(\mathbf{r}, t; \mathbf{r}'', t'')\, \mathbf{v}(\mathbf{r}'', t'')\, \partial'' G(\mathbf{r}'', t''; \mathbf{r}', t' \mid \mathbf{v}(\mathbf{r}', t')) \tag{2.1}$$

where $G^{(0)}(\mathbf{r}, t; \mathbf{r}', t') = G^{(0)}(\mathbf{r} - \mathbf{r}', t - t')$ is the solution of (1.3) for a medium which is at rest $(\mathbf{v}(r, t) = 0)$.

Eq.(2.1) is represented graphically in Fig.1, where Green's function G corresponds to the fat arrow, $G^{(0)}$ corresponds to the thin arrow and $v\partial$ corresponds to the wavy line insertion. The iterative solution of (2.1) is represented graphically by an infinite series (Fig.2). Replacement of the product of the velocities by their statistical moments, which is shown graphically in Fig.3 corresponds to averaging with respect to the velocity ensemble.

In the case of Gaussian statistics, the statistical moments can be represented in the form of the sum of all possible pairwise mean averaged fields (this assertion is referred to as Wick's theorem in field theory). If a velocity pair correlator is denoted graphically by a wavy line with two arrows, we shall have the picture shown in Fig.4 for the mean Green's function. Since the series which have been represented do not correspond to any expansion with respect to a small parameter, it is impossible to confine the iterative series to a finite number of terms and it is necessary to seek a solution of problem (1.3) outside of the framework of perturbation theory, which involves attempts to sum infinite subsequences of a complete series.

Particle by particle summation of the irreducible diagrams (of type b and c in Fig.4) reduces to the solution of Dyson's equation. Taking account of type d diagrams reduces to the insertion of the complete Green's function G instead of the free Green's function $G^{(0)}$ and taking account of diagram e reduces to replacing one of the vertices (the right vertex) by the renormalized vertex. The discarding of type e diagrams (neglecting to renormalize the vertices) corresponds to the approximation of direct interactions /7/. Type d diagrams are summed by the renormalized group method and, when an attempt is made to take account of the renormalization of the vertices it is additionally necessary to consider the equation for the vertex containing the moments of four orders and, once again, one comes up against the problem of closing the chain of equations for the moments. Hence, by summing the series of perturbation theory, it is impossible, when there is no small parameter, to make any successful advances in obtaining rigorous results which leads to the need to give up any attempt at solving the problem within the framework of perturbation theory.

In order to construct a general solution of Eq.(1.3) outside of the framework of perturbation theory, use can be made of a method which has been developed in the theory of stochastic processes and is based on a statistical approach to finding the solution of the diffusion equation and the application of a continuous transformation of the measure of a Wiener process which eliminates the convective term /8/. However, this problem can be solved using the Feynman operator formalism /9/ which offers greater possibilities compared with the methods which have been specially developed solely for the investigation of diffusion processes. The Feynman operator formalism is a generalization of the Fock representation for an inverse operator to the case when there are non-commutating quantities in the operator.

According to Fock, Green's function for the operator $L(\partial_t, \partial)$ can be written in the form

$$G(\mathbf{r}, t;\ \mathbf{r}', t') = L^{-1}(\partial_t, \partial)\, \delta(\mathbf{r} - \mathbf{r}')\, \delta(t - t')$$

$$L^{-1}(\partial_t, \partial) = \int_0^\infty d\tau \exp[-L(\partial_t, \partial)\tau]$$

Fig.1

Fig.2

Fig.3

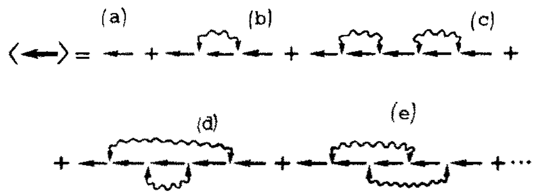

Fig.4

However, in the case under consideration, the operator L depends explicitly on the coordinates and time and, as a consequence of the non-commution of the quantities occurring in it,

$$[\partial_t, \mathbf{v}(\mathbf{r}, t)]_- = \partial_t \mathbf{v}(\mathbf{r}, t), \quad [\partial_i, \mathbf{v}(\mathbf{r}, t)] = \partial_i \mathbf{v}(\mathbf{r}, t)$$

it is necessary to determine the sequence in which the operators in the exponential factor act. For this purpose, following Feynman /9/, we introduce an ordering parameter s and determine the sequence in which the operators $\partial_t(s)$, $\partial(s)$, $\mathbf{v}(\mathbf{r}, t; s)$ act in order of increasing s (of "natural time"). Here, the inverse operator L^{-1} is written in the form* (*E.V. Teodorovich, On infrared divergences and the role of local and non-local interactions in the formation of a state of widespread turbulence: Preprint 388, Inst. Problem Mekhan. Akad. Nauk SSSR, Moscow, 22, 1989).

$$L^{-1}(\partial_t, \partial \,|\, \mathbf{v}(\mathbf{r}, t)) = \int_0^\infty d\tau \exp\left\{-\int_0^\tau ds\,[\partial_t(s) + \mathbf{v}(\mathbf{r}, t; s)\,\partial(s) - \varkappa\partial^2(s)]\right\} \tag{2.2}$$

In order to "disentangle the operator exponent" it is necessary to eliminate the second power of the operator $\partial(s)$ in the exponential factor and it then becomes possible to interpret expressions of the type of $\exp(a\partial_t)$ and $\exp(\mathbf{b}\partial)$ as operators which shift the arguments in accordance with the identities

$$\exp(a\partial_t)\, f(\mathbf{r}, t) = f(\mathbf{r}, t + a), \quad \exp(\mathbf{b}\partial)\cdot f(\mathbf{r}, t) = f(\mathbf{r} + \mathbf{b}, t) \tag{2.3}$$

For this purpose, we apply the Stratonovich transform /10/ which is the functional analogue of a Weierstrass transform /11/,

$$\exp\left[a\int_0^\tau ds\mathbf{f}^2(s)\right] = \int \frac{d[\boldsymbol{\zeta}]}{A(a, \tau)} \exp\left\{-\int_0^\tau ds\left[\frac{\boldsymbol{\zeta}^2(s)}{4a} - \mathbf{f}(s)\,\boldsymbol{\zeta}(s)\right]\right\} \tag{2.4}$$

$$A(a, \tau) = \int d[\boldsymbol{\zeta}] \exp\left\{-\int_0^\tau ds\,\frac{\boldsymbol{\zeta}^2(s)}{4a}\right\}$$

Here $\boldsymbol{\zeta}(s)$ are arbitrary functions which are defined in the interval $(0, \tau)$ and $d[\boldsymbol{\zeta}]$ is an integral measure (an elementary volume in the functional space).

As a result of the application of formulae (2.4) and (2.2) and the use of formulae (2.2) and taking account of the rules for the sequence in which non-commuting operators act, we find

$$L^{-1}(\partial_t, \partial \,|\, \mathbf{v}(\mathbf{r}, t)) = \int_0^\infty \frac{d\tau}{A(\varkappa, \tau)} \int d[\boldsymbol{\zeta}] \times \tag{2.5}$$

$$\exp\left\{-\frac{1}{4\varkappa}\int_0^\tau ds\,[\zeta(s)+\mathbf{V}(\mathbf{r},t,\tau,s;\zeta)]^2\exp\left\{-\int_0^\tau ds\,[\partial_t(s)-\zeta(s)\,\partial(s)]\right\}\right.$$

$$\mathbf{V}(\mathbf{r},t,\tau,s;\zeta)=\mathbf{v}\left(\mathbf{r}+\int_s^\tau ds'\zeta(s'),t-\tau+s\right)$$

After the arguments have been shifted in formula (2.5), no non-commuting operators remain and, as a result of this, it is possible to omit the dependence of $\mathbf{v}(\mathbf{r},t;s)$ on the "natural time" s.

When $\mathbf{v}=0$, formula (2.5) is identical with the Wiener representation for the solution of the diffusion equation in the form of an integral along trajectories. Here, the functional variable $\zeta(s)$, which has been formally introduced in carrying out the Stratonovich transformation, is identical with the velocity of the motion along a trajectory. The generalization to the case when $\mathbf{v}\neq 0$ reduces to making allowance for the inhomogeneous deflection of the trajectories due to the motion of the medium, which corresponds to the assumption which was made without sufficient grounds in /12/.

In can be seen from (2.5) that $(\mathbf{r}-\mathbf{r}',\ t-t')$ and (r,t) are the arguments of Green's function and the dependence on the latter arguments only occurs in terms of the function $\mathbf{v}(\mathbf{r},t)$. It is therefore useful to carry out a Fourier transformation with respect to the variables $\mathbf{r}-\mathbf{r}',\ t-t'$ and then again to use a Stratonovich transformation (2.4) in order to remove the square of the velocity in the exponential factor (2.5). After this transformation has been carried out, we find that the dependence on the velocity $\mathbf{v}(\mathbf{r},t)$ turns out to be isolated in the form of a separate factor. As a result of this, we obtain the equality

$$\langle L^{-1}(-i\omega,i\mathbf{p}\,|\,\mathbf{v}(\mathbf{r},t))\rangle=\overline{G}(\mathbf{p},\omega)=\int_0^\infty\frac{d\tau}{B(\varkappa,\tau)}\int d[\zeta]\,d[\boldsymbol{\eta}]\times$$

$$\exp\left\{-\int_0^\tau ds\,[\varkappa\boldsymbol{\eta}^2(s)-i\omega-i[\mathbf{p}+\boldsymbol{\eta}(s)]\,\zeta(s)]\right\}\times$$

$$\left\langle\exp\left\{i\int_0^\tau ds\boldsymbol{\eta}(s)\,\mathbf{V}(\mathbf{r},t,\tau,s;\zeta)\right\}\right\rangle$$

$$B(\varkappa,\ \tau)=A(\varkappa,\ \tau)\,A(1/(4\varkappa),\ \tau)$$

in averaging over the ensemble, where the average occurring on the right-hand side is the characteristic functional of the velocity field.

In the case of a random Gaussian process, the characteristic functional is expressed in terms of a pair correlation function which, in the case of a homogeneous and stationary stochastic process is solely dependent on the differences in the coordinates and times

$$\langle v_i(\mathbf{r},\ t)\,v_i(\mathbf{r}',\ t')\rangle=C_{ij}(\mathbf{r}-\mathbf{r}',\ t-t')$$

Here, by virtue of the incompressibility condition, it satisfies the equation

$$\partial_i C_{ij}(\mathbf{r}-\mathbf{r}',\ t-t')=\partial_j' C_{ij}(\mathbf{r}-\mathbf{r}',\ t-t')=0$$

The characteristic functional of a Gaussian process is defined by the relationship

$$\left\langle\exp\left\{i\int_0^\tau ds\eta_i(s)\,v_i\left(\mathbf{r}+\int_s^\tau ds'\zeta(s'),t-\tau+s\right)\right\}\right\rangle=\exp\{-w(\tau,0;\boldsymbol{\eta},\zeta)\}$$

$$w(t,t';\boldsymbol{\eta},\zeta)=\frac{1}{2}\int_{t'}^{t}ds\int_{t'}^{t}ds'\eta_i(s)\,\eta_j(s')\,C_{ij}\left(\int_{s'}^{s}d\xi\,\zeta(\xi),s-s'\right)$$

As a result, we find

$$\overline{G}(\mathbf{p},\omega)=\int_0^\infty\frac{d\tau}{B(\varkappa,\tau)}\times \tag{2.6}$$

$$\exp\left\{-\int_0^\tau ds\,[\varkappa\boldsymbol{\eta}^2(s)-i\omega-i[\mathbf{p}+\boldsymbol{\eta}(s)]\,\zeta(s)]-w(\tau,0;\boldsymbol{\eta},\zeta)\right\}$$

for the Fourier transform of the averaged Green function.

We will now present some further formulae which are obtained after carrying out inverse Fourier transformations with respect to the time and coordinates

$$\bar{G}(\mathbf{p}, t-t') = \frac{\theta(t-t')}{B(\varkappa, t-t')} \int d[\zeta]\, d[\eta] \times \tag{2.7}$$
$$\exp\left\{-\int_{t'}^{t} ds\,[\varkappa\eta^2(s) - i\,[\mathbf{p}+\eta(s)]\,\zeta(s)] - w(t, t'; \eta, \zeta)\right\}$$

The formula for $\bar{G}(\mathbf{r}-\mathbf{r}', t-t')$ is obtained from (2.7) if one puts $\mathbf{p}=0$ in the exponential factor and requires that

$$\mathbf{r}-\mathbf{r}' = -\int_{t'}^{t} ds\,\zeta(s)$$

Note that formulae (2.6) and (2.7) are exact within the framework of a model of a statistically homogeneous and stationary Gaussian process for the fluctuations of a velocity field and their derivation is not associated with perturbation theory and any assumption whatsoever concerning the existence of a small parameter. Although continuous integration can only be carried out precisely in certain special cases (see below), formulae (2.6) and (2.7) can serve as a source for obtaining asymptotic expansions or be used in numerical integration. In particular, information can be obtained from (2.6) regarding the effect of large-scale random motions on small-scale motions which is of interest in the theory of turbulence in connection with the problem of separating out the direct interactions of vortices with substantially differing scales which describe transport effects from dynamical interactions which achieve energy transfer along the wave number spectrum (see the reference given in the footnote).

3. In the case when the velocity fluctuations are a random Markov process ("white noise")

$$C_{ij}(\mathbf{r}-\mathbf{r}', t-t') = C_{ij}(\mathbf{r}-\mathbf{r}')\,\delta(t-t') \tag{3.1}$$

it is possible to carry out of all calculations right up to the end since the functional integration in (2.6) can be successfully carried out. For the d-dimensional turbulence we represent the correlation function of the velocity field, taking account of incompressibility, in the form

$$C_{ij}(\mathbf{r}-\mathbf{r}') = \int \frac{d\mathbf{q}}{(2\pi)^d}\left(\delta_{ij} - \frac{q_i q_j}{q^2}\right) C(q)\, e^{i\mathbf{q}(\mathbf{r}-\mathbf{r}')} \tag{3.2}$$

By substituting expression (3.2) into (2.6), and using (3.1) we obtain

$$\bar{G}(\mathbf{p}, \omega) = \int_0^\infty \frac{d\tau}{B(\varkappa, \tau)} \int d[\zeta]\, d[\eta] \exp\left\{-\int_0^\tau ds\,[\varkappa^*\eta^2(s) - i\omega - i[\mathbf{p}+\eta(s)]\,\zeta(s)]\right\} \tag{3.3}$$

$$\varkappa^* = \varkappa + \delta\varkappa = \varkappa + \frac{1}{2}\int \frac{d\mathbf{q}}{(2\pi)^d}\left[1 - \frac{(\zeta\mathbf{q})^2}{\zeta^2 q^2}\right] C(q) \tag{3.4}$$

The integral in (3.4) is independent of the function $\zeta(s)$ which becomes obvious after integration over the angular variables which can be carried out in an explicit form. We have

$$\int d\Omega_d \left[1 - \frac{(\zeta^2\mathbf{q})^2}{\zeta^2 q^2}\right] = \int d\Omega_d \sin^2\theta = s_{d-1}\int_0^\pi \sin^d\theta\, d\theta = \frac{d-1}{d}\, s_d$$

where θ is the angle between the vectors $\zeta(s)$ and $\mathbf{q}$ and s_d is the area of the surface of a d-dimensional sphere of unit radius.

We therefore get

$$\varkappa^* = \varkappa + \frac{d-1}{2d}\,\frac{s_d}{(2\pi)^d}\int_0^\infty q^{d-1} C(q)\, dq \tag{3.5}$$

The functional integrations in (3.3) are carried out using the scheme:

$$\bar{G}(\mathbf{p}, \omega) = \int_0^\infty \frac{d\tau}{B(\varkappa, \tau)} \exp(i\omega\tau) \int d[\zeta] \exp\left\{-\int_0^\tau ds\left[\frac{\zeta^2(s)}{4\varkappa} + \mathbf{p}\zeta(s)\right]\right\} \times$$
$$\int d[\eta] \exp\left\{-\int_0^\tau ds\varkappa^*\left[\eta(s) - \frac{i}{2\varkappa}\zeta(s)\right]^2\right\} =$$

$$\int_0^\infty \frac{d\tau}{B(\varkappa,\tau)} \exp[-(-i\omega+\varkappa^* \mathbf{p}^2)\tau] \int d[\zeta] \exp\left\{-\frac{1}{4\varkappa^*}\int_0^\tau ds\,[\zeta(s)+2\varkappa\mathbf{p}]^2\right\} =$$

$$\int_0^\infty d\tau \frac{B(\varkappa^*,\tau)}{B(\varkappa,\tau)} \exp[-(-i\omega+\varkappa^* p^2)] = [-i\omega+\varkappa^* p^2]^{-1}$$

where the following operations are successively carried out: reduction of the integral to Gaussian form with respect to η, integration with respect to η and reduction of the integral to a Gaussian form with respect to ξ, integration with respect to ζ and application of the relationship $A(\varkappa,\tau)A(\alpha^2/\varkappa,\tau) = A^2(\alpha,\tau)$ which is readily verified for Gaussian integrals.

Hence, in the case of a Markov random process (this assumption means that the displacement of the trajectories at a given point is independent of the shape of the trajectory in the preceding segments which is determined by the overall action of the molecular and turbulent motion), the effect of turbulent mixing reduces to renormalization of the molecular diffusion coefficient, that is, to the replacement $\varkappa \to \varkappa^* = \varkappa + \delta\varkappa$ which corresponds to a faster course for the diffusion processes. Taylor's hypothesis /13/ concerning the additivity of the molecular and turbulent transport coefficients is confirmed in the case of a Markov process.

However, the random fluctuations of a velocity field are not, in fact, Markov processes and, in the general case, functional integration cannot be successfully carried out. Nevertheless, there is one further special case when it can be done, that is, the case of spatially homogeneous random velocities $\mathbf{v}(\mathbf{r},t) = \mathbf{v}(t)$. Here, from formula (2.7) we find

$$G(\mathbf{p},\omega\,|\,\mathbf{v}(t)) = \int_0^\infty d\tau \exp\left\{-\int_0^\tau ds\,[-i\omega+\varkappa p^2+i\mathbf{p}\mathbf{v}(t-\tau+s)]\right\} \tag{3.6}$$

after functional integration with respect to ζ.

Averaging over the ensemble of random velocity samples we obtain

$$\langle G(\mathbf{p},\omega\,|\,\mathbf{v}(t))\rangle = \int_0^\infty d\tau \exp[-(-i\omega+\varkappa p^2)]\,W(\mathbf{p},t,\tau) \tag{3.7}$$

where

$$W(\mathbf{p},t,\tau) = \left\langle \exp\left[-i\int_0^\tau ds\,\mathbf{p}\mathbf{v}(t-\tau+s)\right]\right\rangle$$

is the characteristic functional of the velocity field. In the case of a random Gaussian process

$$W(\mathbf{p},t,\tau) = \exp\left\{-\frac{1}{2}\int_0^\tau ds\int_0^\tau ds'\,\langle \mathbf{p}\mathbf{v}(t-\tau+s)\cdot\mathbf{p}\mathbf{v}(t-\tau+s')\rangle\right\}$$

In connection with formula (3.6), we note that it is identical with the expression for Green's function in the transport approximation*. (*V.I. Belinicher and V.S. L'vov, Scale-invariant theory of widespread hydrodynamic turbulence: Prepint 333, Inst. Avtomatiki i Elektrometrii, Sib. Otd. Akad. Nauk SSSR, Novosibirsk, 1986.)

In the case of a statistically stationary and isotropic velocity field, we have

$$\langle v_j(t)\,v_j(t')\rangle = \delta_{ij}C(t-t')$$

and, as the result after carrying out an inverse Fourier transformation, we find

$$\bar{G}(\mathbf{r},t) = \frac{\theta(t)}{[4\pi\varkappa t^*]^{d/2}} \exp\left\{-\frac{r^2}{4\varkappa t^*}\right\}$$
$$t^* = t + \frac{1}{2\varkappa}\int_0^t ds\int_0^t ds'\,C(s-s') \tag{3.8}$$

Hence, in the problem under consideration, the evolution of the concentration occurs both as in the case of conventional diffusion processes but with the replacement of the time by an effective time t^* which is determined by the second relationship of (3.8). In the case of a Markov process $C(t-t') = C_0\delta(t-t')$, the effective time is proportional to the conventional time:

$$t^* = [1 + C_0/(2\varkappa)]\, t$$

and the result to a renormalization of the diffusion coefficient $\varkappa \to \varkappa^* = \varkappa + {}^1/_2 C_0$ as in the case of inhomogeneous Markov velocity pulsations.

Although turbulent velocity pulsations are not, in fact, Markov processes /6, 14/, the Markov approximation must nevertheless yield sensible results when estimating the influence of small-scale velocity pulsations on large-scale processes since

$$C_{ij}(\mathbf{q}, t-t') = C_{ij}(\mathbf{q}) \exp[-\nu(q) q^2 \mid t-t' \mid]$$

and, for large q, the correlation time will be small. The estimation of the corrections to the Markov approximation in the diffusion problem has been considered by Drummond /12/.

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